Multiples of 2, 3 and 5

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Smoking Gun Interactive

Abstract

This paper presents the solution to the problem of “Determine the nth multiple of only 2, 3 or 5”. It is divided on several sections, first, the problem is introduced, then the proposed solution is explained and proven. I conclude by highlighting points of improvements.

Keywords: multiples, nth multiple, prime numbers

**Multiples of 2, 3 and 5 – Problem Description**

Consider a series in ascending order that only consists of numbers that can be factored by any combination of and . e.g.

For example, the numbers (prime), (prime) or ( is not a valid factor), are not in the above series. They are not factorable by .

The number is included.

For example, the number in position would be :

**Question**: Design an algorithm to find the number that occupies position in this series.   
*NOTE*: the correct answer is , use this to verify your algorithm.

**Delivery Method:**

The solution is delivered under the Multiples/ directory.

The dist/ folder contains the deliverable objects, composed of:

* A Windows Console application (Multiples.UI.exe), ready to run. It executes the service for the stated factors () and iterates indefinitely asking the user to enter the position they wish to calculate.
* The final signed version of this document (Multiples of 2, 3 and 5.pdf)

The src/ folder contains the source code used to build the solution:

* The implementation of the solution is delivered as a service located in a .NET Standard Library named /Multiples.Core.
* The service is presented to the user as a Windows Console Application named /Multiples.UI that executes the service for the stated factors () and iterates indefinitely asking the user to enter the position they wish to calculate.
* Additionally, an xUnit Test project named /Multiples.Core.Tests is provided to extensively test the provided service.
* The Word version of this document (Multiples of 2, 3 and 5.docx)

**Proposed Solution**

The proposed solution solves a generalization of the stated problem to be applied for any given list of factors (besides 2, 3 and 5)

The solution computes all the valid multiples for the given factors until the multiple in the requested position is found. The operational complexity of the solution linearly depends on the position of the multiple to compute.

The core idea of the solution is that every multiple of the list must be factored only by the provided factors, which means that if is a valid multiple there must be a way to compose it as where is the given factor and .

Example:

1. With factors: 2, 3 and 5. 10 is a valid multiple given than . But 14 is not because the only way to factor it out is as and is not part of the given list.
2. With factors: 3, 4 and 8. is a valid multiple given it can be factored as even when it can also be factored as

The first multiple will always be 1 and it cannot be part of the list.

The 2nd multiple is the smaller provided factor.

To determine the next multiples, we ponder on the idea that if is a valid multiple then is also a valid multiple and we compute the list of every possible multiple by multiplying each of the factors by the previously determined multiples.

To implement the idea, we keep an internal state for each factor which stores a reference to the next already determined multiple and multiplies its values by the factor.

Ex:

At the start, the smallest factor is always the 2nd multiple, so it’s initial state is: while the other factors are initialized to

For every iteration, the next multiple in the list is the smaller .

After a new multiple has been determine, the of the chosen factor (that with the smallest ) gets updated as:

1. if
2. otherwise

When the value in is the desired result.

The computed values remain stored so successive executions of the solution won’t recompute already determined multiples.

## Proof of Correctness:

In order to proof the solution, let’s first proof the following 3 lemmas:

### Lemma #1: “Every element in the list of Multiples is a valid multiple”

We can proof this lemma using the Principle of Mathematic Induction:

1. The 1st multiple () is a valid multiple, following the problem definition.
2. The 2nd multiple is always the smallest factor and , so it’s a valid multiple.
3. We have that and assuming is a valid multiple, then is also a valid multiple, which proofs this lemma.

### Lemma #2: “Every valid multiple, is contained in the list of Multiples”

We can proof this lemma using “Reductio ad absurdum”:

1. Let’s take the smallest which is a valid multiple of but not included in .
2. There’s a way to write it as where is the given factor and by definition of a valid multiple.
3. We know that otherwise it would belong to .
4. cannot be written as , where , otherwise it would belong to .
5. After 3. we know that is composed of at least .
6. After 5. we know that .
7. Let’s say , which, after 5. and a multiple of .
8. After 6. , where, because of 7. is a multiple of contradicting 4 and proving this lemma.

### Lemma #3: “Every element in the list is greater than the elements before it”

We can proof this lemma by using “Reductio ad absurdum”.

1. Let’s take as the smallest multiple with an
2. We know that:
   1. , with
   2. , with
   3. And,
3. When each value was added to the list, its factor status was:
4. was added to the list before by hypothesis.
5. Let’s call , then :
   1. If then , with , otherwise . Furthermore because is a monotonous increasing function.
   2. If then by solution definition implying that given that is a monotonous increasing function and
6. By hypothesis , otherwise wouldn’t be the first multiple lesser than . So:
   1. implies implies implies which contradicts the hypothesis and proves the lemma.

### Proof of Correctness:

1. contains every possible valid multiple of
2. is the valid multiple in the given position.

## Solution Complexity:

### Spatial Complexity:

Proof:

1. We need to compute the list of every multiple until the list size is the same as the required   
   position. At the end of the algorithm .
2. We know that , where is the constant size of .
3. We know that , where is the constant size of each .
4. Every other memory consumed is constant regardless the input so it can be represented as
5. We have that:

### Time Complexity:

Note: For the scope of the problem,

Proof:

1. The execution stops when .
2. Every iteration we increase one by one.
3. will always refer to an existing index in the list:

* Let’s consider the step prior to being increased.
* If implies so a new item gets added to the list and implying that
* If implies

1. After 1., 2. and 3. the maximum amount of iterations will be .
2. Every iteration performs a constant number of operations .
3. As consequence

**Conclusion:**

I couldn’t proof meaning there’s the possibility of a better solution to the problem in the realm of or lower.

Same reasoning applies to .

Considering the scope of the problem with , given that and the main computation limitation becomes the maximum possible representable multiple, which in the chosen framework is the 64-bit unsigned integer primitive type, which can hold up to the value , which makes as the biggest possible position to compute., being

In order to compute larger positions, we would have to resort to the use of possibly making the algorithm impractical.

Considering the relatively smaller maximum () input size, both, the practical and are small enough as to be considered acceptable solution for every scenario other than those that require the maximum possible optimizations.

As consequence, I made the decision to prioritize time to market, considering that further effort investment wouldn’t result in a much higher ROI under most real-life scenarios.   
Further research is recommended to either proof or develop a more efficient solution if such investment is deemed required.